Please check the examination details below before entering your candidate information

| Candidate surname | Other names |
|---|---|
| | |

**Pearson Edexcel Level 3 GCE**

| Centre Number | Candidate Number |
|---|---|
| | |

# Thursday 13 June 2019

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **9FM0/3A**

# Further Mathematics
## Advanced
## Paper 3A: Further Pure Mathematics 1

**You must have:**
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

**Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

### Instructions
- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

### Information
- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

### Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

*Turn over*





Pearson

**Answer ALL questions. Write your answers in the spaces provided.**

1. Use Simpson's rule with 4 intervals to estimate

$$\int_{0.4}^{2} e^{x^2}\,dx$$

**(5)**

$h = \dfrac{(2)-(0.4)}{4} = \dfrac{2}{5}$

$h = \dfrac{b-a}{n}$ for $\int_a^b f(x)\,dx$

n no. of strips

$h = \frac{2}{5}$

| $x$ | 0.4 | 0.8 | 1.2 | 1.6 | 2.0 |
|---|---|---|---|---|---|
| $y$ | $e^{0.16}$ | $e^{0.64}$ | $e^{1.44}$ | $e^{2.56}$ | $e^{4}$ |

$\boxed{\text{Area} = \frac{1}{3}h\left(\text{endpoints} + 4(\text{odds}) + 2(\text{evens})\right)}$

* not given in F.B.
– must learn –

$= \frac{1}{3}\left(\frac{2}{5}\right)\left(\left(e^{0.16}+e^{4}\right) + 4\left(e^{0.64}+e^{2.56}\right) + 2\left(e^{1.44}\right)\right)$

$= \frac{2}{15}\left(123.5422453\right)$

$= 16.4722\,9938$

$\approx 16.5$

2. Given that $k$ is a real non-zero constant and that

$$y = x^3 \sin kx$$

use Leibnitz's theorem to show that

$$\frac{d^5y}{dx^5} = (k^2x^2 + A)k^3x\cos kx + B(k^2x^2 + C)k^2 \sin kx$$

where $A$, $B$ and $C$ are integers to be determined.

**(4)**

$y = \underbrace{x^3}_{u} \underbrace{\sin kx}_{v}$

$u = x^3$ $\quad$ $v = \sin kx$

$u' = 3x^2$ $\quad$ $v' = k\cos kx$

$u'' = 6x$ $\quad$ $v'' = -k^2\sin kx$

$u''' = 6$ $\quad$ $v''' = -k^3\cos kx$

$u^{IV} = 0$ $\quad$ $v^{IV} = k^4\sin kx$

$u^{V} = 0$ $\quad$ $v^{V} = k^5\cos kx$

Leibnitz's Theorem: $\displaystyle\sum_{r=0}^{n} \binom{n}{r} u^{(n-r)} v^{(r)}$

× $\quad$ × $\quad$ (multiply)

$\binom{5}{0} = 1 \longrightarrow u = x^3$ $\quad$ $v = \sin kx$

$\binom{5}{1} = 5 \longrightarrow u' = 3x^2$ $\quad$ $v' = k\cos kx$

$\binom{5}{2} = 10 \longrightarrow u'' = 6x$ $\quad$ $v'' = -k^2\sin kx$

$\binom{5}{3} = 10 \longrightarrow u''' = 6$ $\quad$ $v''' = -k^3\cos kx$

$\binom{5}{4} = 5 \longrightarrow u^{IV} = 0$ $\quad$ $v^{IV} = k^4\sin kx$

$\binom{5}{5} = 1 \longrightarrow u^{V} = 0$ $\quad$ $v^{V} = k^5\cos kx$

$$\frac{d^5y}{dx^5} = x^3k^5\cos(kx) + 15x^2k^4\sin(kx) - 60xk^3\cos(kx) - 60k^2\sin(kx)$$

$$\frac{d^5y}{dx^5} = (x^3k^5 - 60xk^3)\cos kx + (15x^2k^4 - 60k^2)\sin kx$$

$$\frac{d^5y}{dx^5} = (k^2x^2 - 60)k^3x\cos kx + 15(k^2x^2 - 4)k^2\sin kx \;//$$

$A = -60,\ B = 15,\ C = -4$



P 6 1 1 7 9 A 0 4 3 2

**3.**

$$\frac{dy}{dx} = x - y^2 \qquad \text{(I)}$$

(a) Show that

$$\frac{d^5y}{dx^5} = ay\frac{d^4y}{dx^4} + b\frac{dy}{dx}\frac{d^3y}{dx^3} + c\left(\frac{d^2y}{dx^2}\right)^2$$

where $a$, $b$ and $c$ are integers to be determined.

**(4)**

(b) Hence find a series solution, in ascending powers of $x$ as far as the term in $x^5$, of the differential equation (I), given that $y = 1$ at $x = 0$

**(5)**

a. $\dfrac{dy}{dx} = x - y^2$

differentiate w.r.t. $x$.

product rule $\begin{cases} y = uv \\ y = uv' + u'v \end{cases}$

$$\frac{d^2y}{dx^2} = 1 - 2y\left(\frac{dy}{dx}\right)$$

chain rule $\begin{cases} y = (u)^n \\ y = n(u)^{n-1}(u') \end{cases}$

$$\frac{d^3y}{dx^3} = -2y\left(\frac{d^2y}{dx^2}\right) - 2\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3} = -2y\left(\frac{d^2y}{dx^2}\right) - 2\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^4y}{dx^4} = -2\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right) - 2y\left(\frac{d^3y}{dx^3}\right) - 4\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$$

$$\frac{d^5y}{dx^5} = -2\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2y}{dx^2}\right) - 2\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right) - 2\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right) - 2y\left(\frac{d^4y}{dx^4}\right) - 4\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2y}{dx^2}\right) - 4\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right)$$

$$\frac{d^5y}{dx^5} = -2y\left(\frac{d^4y}{dx^4}\right) - 2\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right) - 2\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right) - 4\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right) - 2\left(\frac{d^2y}{dx^2}\right)^2 - 4\left(\frac{d^2y}{dx^2}\right)^2$$

rearrange into Q form

$$\frac{d^5y}{dx^5} = -2y\left(\frac{d^4y}{dx^4}\right) - 8\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right) - 4\left(\frac{d^2y}{dx^2}\right)^2 \;//$$

$a = -2,\ b = -8,\ c = -4$





P 6 1 1 7 9 A 0 6 3 2

**Question 3 continued**

b. $x_0 = 0, \quad y_0 = 1$

$$\left.\frac{dy}{dx}\right|_{x=0,y=1} \quad (0) - (1)^2 = -1$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0,y=1} \quad 1 - 2(1)(-1) = 3$$

$$\left.\frac{d^3y}{dx^3}\right|_{x=0,y=1} \quad -2(-1)^2 - 2(1)(3) = -8$$

$$\left.\frac{d^4y}{dx^4}\right|_{x=0,y=1} \quad -4(-1)(3) - 2(-1)(3) - 2(1)(-8) = 34$$

$$\left.\frac{d^5y}{dx^5}\right|_{x=0,y=1} \quad -6(3)^2 - 8(-1)(-8) - 2(1)(34) = -186$$

$$y = y_0 + x\left.\frac{dy}{dx}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{x=0} + \frac{x^3}{3!}\left.\frac{d^3y}{dx^3}\right|_{x=0} + \frac{x^4}{4!}\left.\frac{d^4y}{dx^4}\right|_{x=0} + \frac{x^5}{5!}\left.\frac{d^5y}{dx^5}\right|_{x=0} + \dots$$

$$y = 1 + x(-1) + \frac{x^2}{2}(3) + \frac{x^3}{6}(-8) + \frac{x^4}{24}(34) + \frac{x^5}{120}(-186)$$

$$y = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5 + \dots \quad //$$

**4.** The parabola $C$ has equation

$$y^2 = 16x$$

The distinct points $P(p^2, 4p)$ and $Q(q^2, 4q)$ lie on $C$, where $p \neq 0$, $q \neq 0$

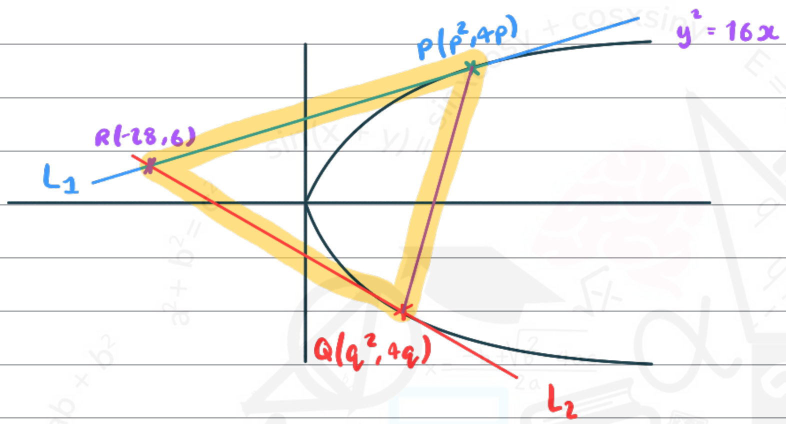
The tangent to $C$ at $P$ and the tangent to $C$ at $Q$ meet at the point $R(-28, 6)$.

Show that the area of triangle $PQR$ is 1331

**(8)**

$y^2 = 16x$

$y^2 = 4(4)x \qquad a = 4$




$y^2 = 16x$ differentiate both sides w.r.t. $x$

$2y\left(\frac{dy}{dx}\right) = 16$

$\frac{dy}{dx} = \frac{16}{2y} = \frac{8}{y}$

$\left.\frac{dy}{dx}\right|_{y=4p} \quad \frac{8}{4p} = \frac{2}{p}$

$M_{tangent} = \frac{2}{p}$

$y - (4p) = \frac{2}{p}(x - p^2)$

$py - 4p^2 = 2(x - p^2)$

$py - 4p^2 = 2x - 2p^2$

$py = 2x + 2p^2 \qquad (L_1)$

$(-28, 6)$ lies on this line so sub in to find $p$

$p(6) = 2(-28) + 2p^2$

$2p^2 - 6p - 56 = 0$

**Question 4 continued**

$p^2 - 3p - 28 = 0$

$(p+4)(p-7) = 0$

$p = -4$ or $p = 7$

$\left.\frac{dy}{dx}\right|_{y=4q} \quad \frac{8}{4q} = \frac{2}{q}$

$m_{tangent} = \frac{2}{q}$

$y - (4q) = \frac{2}{q}(x - q^2)$

$qy - 4q^2 = 2(x - q^2)$

$qy - 4q^2 = 2x - 2q^2$

$qy = 2x + 2q^2 \qquad (L_2)$

$(-28, 6)$ lies on this line so sub in to find p

$q(6) = 2(-28) + 2q^2$

$2q^2 - 6q - 56 = 0$

$q^2 - 3q - 28 = 0$

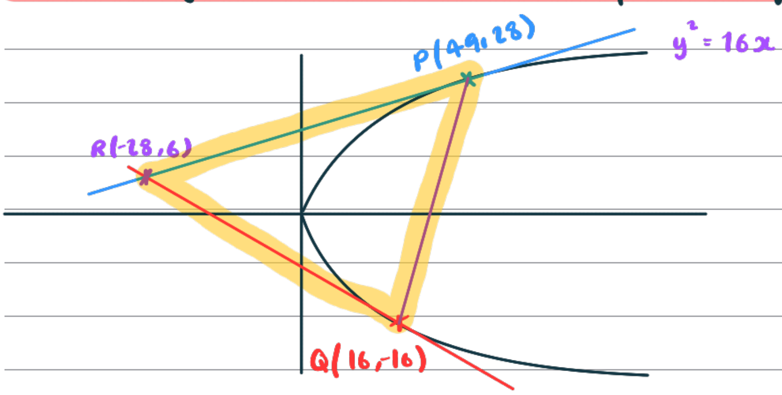
$(q-7)(q+4) = 0$

$q = -4$ or $q = 7$

either $p = -4$ and $q = 7$
or
$p = 7$ and $q = -4$

doesn't matter which sol$^n$ you pick.

For our diagram to make sense, let's pick $q = -4$, $p = 7$.




Coord. P: $(49, 28)$

Coord. Q: $(16, -16)$

**Question 4 continued**

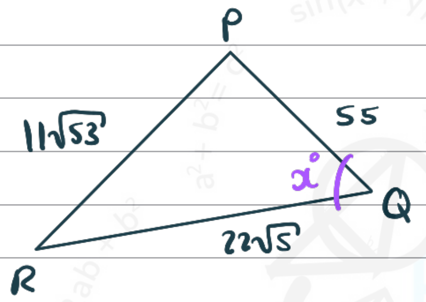
Must find all lengths, $|QR|$, $|PQ|$, $|PR|$.

Then calculate any angle, and then calculate $\Delta$ Area of Triangle ($\frac{1}{2}ab\sin\hat{c}$)

$|PR| = \sqrt{(49+28)^2 + (28-6)^2} = 11\sqrt{53}$

$|PQ| = \sqrt{(49-16)^2 + (28+16)^2} = 55$

$|QR| = \sqrt{(16+28)^2 + (-16-6)^2} = 22\sqrt{5}$




Let's work out $\angle PQR$.

$\cos(\angle PQR) = \dfrac{(22\sqrt{5})^2 + (55)^2 - (11\sqrt{53})^2}{2(22\sqrt{5})(55)}$

$\cos(\angle PQR) = -0.178885438$

$\angle PQR = \cos^{-1}(-0.178885438)$

$\angle PQR = 100.3048465°$

$\Delta PQR = \frac{1}{2}(22\sqrt{5})(55)\sin(100.3048465)$ use $\frac{1}{2}ab\sin\hat{c} = \Delta$ AOT

$\Delta PQR = 1331$ units$^2$

5.

$$I = \int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x \qquad 0 < x < \frac{\pi}{4}$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \frac{1}{5}\ln\left(\frac{2+\tan\left(\frac{x}{2}\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + k$$

where $k$ is an arbitrary constant.

**(8)**

$t = \tan\left(\frac{x}{2}\right)$

$\sin(x) = \frac{2t}{1+t^2}$

$\cos(x) = \frac{1-t^2}{1+t^2}$

$\tan(x) = \frac{2t}{1-t^2}$

proof @ end of Q.

$I = \int \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)}\,dx$

$I = \int \frac{1}{\frac{4(1-t^2)-3(2t)}{1+t^2}}\,dx$

$I = \int \frac{1+t^2}{-4t^2-6t+4}\,dx$

$t = \tan\left(\frac{x}{2}\right)$

$x = 2\arctan(t)$

$\frac{dx}{dt} = 2\left(\frac{1}{1+t^2}\right) = \frac{2}{1+t^2}$

$dx = \frac{2}{1+t^2}\,dt$

must change $dx$ to $dt$.

**Question 5 continued**

$$I = \int \frac{\cancel{1+t^2}}{-4t^2-6t+4} \times \frac{2}{\cancel{1+t^2}}\, dt$$

$$I = \int \frac{\cancel{2}}{\cancel{2}(-2t^2-3t+2)}\, dt$$

$$I = \int \frac{1}{-2t^2-3t+2}\, dt$$

$$\int \frac{1}{-2\left[t^2+\frac{3}{2}t\right]+2}\, dt$$

$$\int \frac{1}{-2\left[\left(t+\frac{3}{4}\right)^2-\frac{9}{16}\right]+2}\, dt$$

$$\int \frac{1}{-2\left(t+\frac{3}{4}\right)^2+\frac{9}{8}+2}\, dt$$

$$\int \frac{1}{-2\left(t+\frac{3}{4}\right)^2+\frac{25}{8}}\, dt$$

$$\int \frac{1}{\frac{25}{8}-2\left(t+\frac{3}{4}\right)^2}\, dt$$

Complete the square for quadratic on denominator

$$\frac{1}{2}\int \frac{1}{\frac{25}{16}-\left(t+\frac{3}{4}\right)^2}\, dt$$

Is in form:

| $f(x)$ | $f'(x)$ | |
|---|---|---|
| $\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\|$ | $\frac{1}{a^2-x^2}$ | $a>0$ |

Given in F.B.

$a^2 = \frac{25}{16} \Rightarrow a = +\frac{5}{4}$

**Question 5 continued**

let $x = t + 3/4$

$$I = \frac{1}{2}\left[\frac{1}{2\left(\frac{5}{4}\right)} \ln\left|\frac{\frac{5}{4} + \left(t + \frac{3}{4}\right)}{\frac{5}{4} - \left(t + \frac{3}{4}\right)}\right|\right]$$

$$I = \frac{1}{5} \ln\left|\frac{t+2}{\frac{1}{2} - t}\right| + c$$

$$I = \frac{1}{5} \ln\left|\frac{2 + \tan\left(\frac{x}{2}\right)}{\frac{1}{2} - \tan\left(\frac{x}{2}\right)}\right| + c$$

$$I = \frac{1}{5} \ln\left|\frac{2\left(2 + \tan\left(\frac{x}{2}\right)\right)}{2\left(\frac{1}{2} - \tan\left(\frac{x}{2}\right)\right)}\right| + c$$

$$I = \frac{1}{5} \ln\left|\frac{(2)\left(2 + \tan\left(\frac{x}{2}\right)\right)}{1 - 2\tan\left(\frac{x}{2}\right)}\right| + c$$

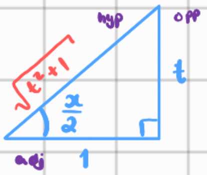
$$= \frac{1}{5} \ln\left|\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2\tan\left(\frac{x}{2}\right)}\right| + \frac{1}{5}\ln(2) + c$$

(use log laws: $\ln(ab) = \ln(a) + \ln(b)$)

$$= \frac{1}{5} \ln\left|\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2\tan\left(\frac{x}{2}\right)}\right| + K \quad //$$

(shown)

# Deriving the t-formulae :




1) draw a right-angled triangle and label angle and sides which you know.

> ✱ you are allowed to memorise the t-formulae for sin(x), cos(x), tan(x) and do not have to derive it in the exam unless specifically asked.

2) work out hypotenuse in terms of t, (pythagoras)

$$\sqrt{(t)^2+(1)^2}$$
$$= \sqrt{t^2+1}$$

3) label each side of triangle opposite, adjacent, hypotenuse

4) write out $\sin\left(\frac{x}{2}\right), \cos\left(\frac{x}{2}\right), \tan\left(\frac{x}{2}\right)$ in terms of t.

$$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{t^2+1}} \qquad \sin = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{t^2+1}} \qquad \cos = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{t}{1} = t \qquad \tan = \frac{\text{opposite}}{\text{adjacent}} \Big/ \frac{\sin}{\cos}$$

5) Now use double-angle formulae and write in terms of t :

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$
$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\sin(x) = 2\left(\frac{t}{\sqrt{t^2+1}}\right)\left(\frac{1}{\sqrt{t^2+1}}\right) \qquad \cos(x) = \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2$$

$$\sin(x) = \frac{2t}{t^2+1} \qquad \cos(x) = \frac{1-t^2}{1+t^2}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{2t}{\cancel{1+t^2}}}{\frac{1-t^2}{\cancel{1+t^2}}} = \frac{2t}{1-t^2}$$

$$\tan(x) = \frac{2t}{1-t^2}$$

6. The concentration of a drug in the bloodstream of a patient, $t$ hours after the drug has been administered, where $t \leqslant 6$, is modelled by the differential equation

$$t^2\frac{d^2C}{dt^2} - 5t\frac{dC}{dt} + 8C = t^3 \qquad \text{(I)}$$

where $C$ is measured in micrograms per litre.

(a) Show that the transformation $t = e^x$ transforms equation (I) into the equation

$$\frac{d^2C}{dx^2} - 6\frac{dC}{dx} + 8C = e^{3x} \qquad \text{(II)}$$

(5)

(b) Hence find the general solution for the concentration $C$ at time $t$ hours.

(7)

Given that when $t = 6$, $C = 0$ and $\frac{dC}{dt} = -36$

(c) find the maximum concentration of the drug in the bloodstream of the patient.

(5)

a. $t = e^x$

$\frac{dt}{dx} = e^x = t$

$\frac{dC}{dt} = \frac{dC}{dx} \times \frac{dx}{dt}$ ← can rewrite $\frac{dC}{dt}$ in terms of product rule

$\frac{dC}{dt} = \frac{1}{t}\frac{dC}{dx} = t^{-1}\frac{dC}{dx}$

differentiate both sides w.r.t. $t$.

$\frac{d^2C}{dt^2} = -t^{-2}\frac{dC}{dx} + t^{-1}\left(\frac{d^2C}{dx^2}\right)\left(\frac{dx}{dt}\right)$

$t^2\left(-\frac{1}{t^2}\left(\frac{dC}{dx}\right) + \frac{1}{t}\left(\frac{d^2C}{dx^2}\right)\left(\frac{1}{t}\right)\right) - 5t\left(\frac{1}{t}\frac{dC}{dx}\right) + 8C = (e^x)^3$

Sub in $\frac{dC}{dt}$ and $\frac{d^2C}{dt^2}$ new eq$^n$s to get a new eq$^n$ in terms of $\frac{dC}{dx}$.

$-\frac{dC}{dx} + \frac{d^2C}{dx^2} - 5\frac{dC}{dx} + 8C = e^{3x}$

$\frac{d^2C}{dx^2} - 6\frac{dC}{dx} + 8C = e^{3x}$ // (shown)

**Question 6 continued**

b. $m^2 - 6m + 8 = 0$ (solve auxillary eq$^n$)

$(m-2)(m-4) = 0$

$m = 2$ or $m = 4$

$C = Ae^{2x} + Be^{4x}$

pick particular integral (P.I.) = $\lambda e^{3x}$

let $C = \lambda e^{3x}$

$C' = 3\lambda e^{3x}$

$C'' = 9\lambda e^{3x}$

$(9\lambda e^{3x}) - 6(3\lambda e^{3x}) + 8(\lambda e^{3x}) = e^{3x}$

$9\lambda e^{3x} - 18\lambda e^{3x} + 8\lambda e^{3x} = e^{3x}$

$-\lambda e^{3x} = e^{3x}$

$-\lambda = 1$

$\lambda = -1$

$C = Ae^{2x} + Be^{4x} - e^{3x}$

gen sol$^n$ = Aux. eq$^n$ + C.F.

$C = A(e^x)^2 + B(e^x)^4 - (e^x)^3$

↓ convert back in terms of t.

$C = At^2 + Bt^4 - t^3$ //

c. $t = 6$ $C = 0$ $\frac{dC}{dt} = -36$

$C = At^2 + Bt^4 - t^3$

$\frac{dC}{dt} = 2At + 4Bt^3 - 3t^2$

$0 = A(6)^2 + B(6)^2 - (6)^3$

$(6)^3 = 36A + 36B$

$216 = 36A + 36B \iff A + B = 6$ (1)

**Question 6 continued**

$-36 = 2A(6) + 4B(6)^3 - 3(6)^2$

$-36 = 12A + 864B - 108$

$12A + 864B = 72$ (2)

$A + B = 6$

$12A + 864B = 72$

$12A + 12B = 72 \Longleftarrow$ (1) $\times 12$

$12A + 864B = 72$ (−)

$-852B = 0$

$-B = 0$ $\boxed{B = 0}$

$A + (0) = 6$

$\boxed{A = 6}$

$C = 6t^2 + \cancel{0t^4} - t^3$

$\boxed{C = 6t^2 - t^3}$

max. conc. @ $\frac{dc}{dt} = 0$

$\frac{dc}{dt} = 12t - 3t^2$

$12t - 3t^2 = 0$

$3t(4 - t) = 0$

$t = 0$ or $t = 4$

@ $t = 0$ $C = 6(0)^2 - (0)^3 = 0$

@ $t = 4$ $C = 6(4)^2 - (4)^3 = \underline{\underline{32}}$

can also check max. conc. from $\frac{d^2c}{dt^2}$, $\frac{d^2c}{dt^2} < 0$ if max. conc.

$\frac{d^2c}{dt^2} = 12 - 6t$ $\quad \left.\frac{d^2c}{dt^2}\right|_{t=4}$ $12 - 6(4) = -12$ $\quad \frac{d^2c}{dt^2} < 0$ $\therefore$ maximum

$\boxed{\text{max. conc.} = 32\text{mg/L}}$ //

**7.** With respect to a fixed origin $O$, the points $A$, $B$ and $C$ have coordinates (3, 4, 5), (10, –1, 5) and (4, 7, –9) respectively.

The plane $\Pi$ has equation $4x - 8y + z = 2$

The line segment $AB$ meets the plane $\Pi$ at the point $P$ and the line segment $BC$ meets the plane $\Pi$ at the point $Q$.

(a) Show that, to 3 significant figures, the area of quadrilateral $APQC$ is 38.5

**(6)**

The point $D$ has coordinates $(k, 4, -1)$, where $k$ is a constant.

Given that the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ form three edges of a parallelepiped of volume 226

(b) find the possible values of the constant $k$.

**(4)**

a. find line eq$^n$ AB:

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -4 \\ -5 \end{pmatrix} + \begin{pmatrix} 10 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix}$

line eq$^n$ AB: $r = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -5 \\ 0 \end{pmatrix} \iff r = \begin{pmatrix} 3+7\lambda \\ 4-5\lambda \\ 5 \end{pmatrix}$

$\Pi$ eq$^n$: $r \cdot \begin{pmatrix} 4 \\ -8 \\ 1 \end{pmatrix} = 2$

sub in line eq$^n$ into r $\Pi$ eq$^n$.

$\begin{pmatrix} 3+7\lambda \\ 4-5\lambda \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -8 \\ 1 \end{pmatrix} = 2$

$4(3+7\lambda) + (-8)(4-5\lambda) + (5)(1) = 2$

$12 + 28\lambda - 32 + 40\lambda + 5 = 2$

$68\lambda = 17$

$\lambda = 1/4$ sub back into gen eq$^n$ to find P.





P 6 1 1 7 9 A 0 2 4 3 2

**Question 7 continued**

$$r = \begin{pmatrix} 3+7(1/4) \\ 4-5(1/4) \\ 5 \end{pmatrix} = \begin{pmatrix} 19/4 \\ 11/4 \\ 5 \end{pmatrix}$$

point P: $\begin{pmatrix} 19/4 \\ 11/4 \\ 5 \end{pmatrix}$

find line eq$^n$ BC:

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$\overrightarrow{BO} = \begin{pmatrix} -10 \\ 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \\ -9 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \\ -14 \end{pmatrix}$

line eq$^n$ BC: $r = \begin{pmatrix} 4 \\ 7 \\ -9 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 8 \\ -14 \end{pmatrix} \iff r = \begin{pmatrix} 4-6\mu \\ 7+8\mu \\ -9-14\mu \end{pmatrix}$

Sub in line eq$^n$ into r $\Pi$ eq$^n$.

$$\begin{pmatrix} 4-6\mu \\ 7+8\mu \\ -9-14\mu \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -8 \\ 1 \end{pmatrix} = 2$$

$4(4-6\mu) + (-8)(7+8\mu) + (1)(-9-14\mu) = 2$

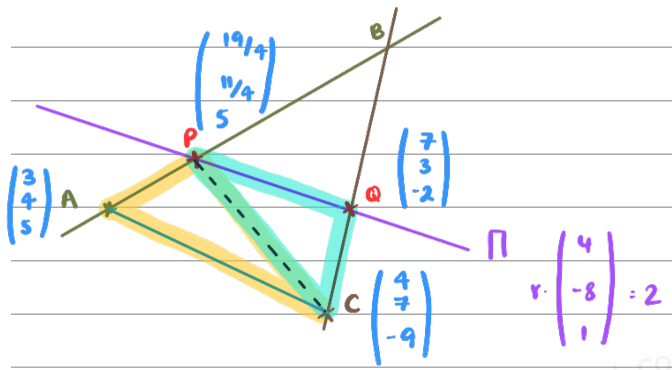
$16 - 24\mu - 56 - 64\mu - 9 - 14\mu = 2$

$-102\mu = 51$

$\mu = -1/2$ Sub back into gen eq$^n$ to find Q.

$$r = \begin{pmatrix} 4-6(-1/2) \\ 7+8(-1/2) \\ -9-14(-1/2) \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} \qquad Q: \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix}$$

**Question 7 continued**




① ②

$\Delta APC + \Delta PQC = \square APQC$

①

$\Delta APC = \frac{1}{2}\left|\vec{AP} \times \vec{AC}\right|$

$\vec{AP} = \vec{AO} + \vec{OP}$

$$\vec{AP} = \begin{pmatrix} -3 \\ -4 \\ -5 \end{pmatrix} + \begin{pmatrix} 19/4 \\ 11/4 \\ 5 \end{pmatrix} = \begin{pmatrix} 7/4 \\ -5/4 \\ 0 \end{pmatrix}$$

$\vec{AC} = \vec{AO} + \vec{OC}$

$$\vec{AQ} = \begin{pmatrix} -3 \\ -4 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -14 \end{pmatrix}$$

$$\vec{AP} \times \vec{AC} = \begin{array}{ccc} i & j & k \\ \hline \frac{7}{4} & -\frac{5}{4} & 0 \\ 1 & 3 & -14 \end{array} = i\left[\left(-\tfrac{5}{4}\right)(-14) - (0)(3)\right] - j\left[\left(\tfrac{7}{4}\right)(-14) - (0)(1)\right] + k\left[\left(\tfrac{7}{4}\right)(3) - (1)\left(-\tfrac{5}{4}\right)\right]$$

given in F.B.

$$= \frac{35}{2}i - \frac{49}{2}j + \frac{13}{2}k$$

$$\Delta APC = \frac{1}{2}\left|\begin{pmatrix} 35/2 \\ -49/2 \\ 13/2 \end{pmatrix}\right|$$

**Question 7 continued**

$\Delta APC = \frac{1}{2}\sqrt{(35/2)^2 + (-\frac{49}{2})^2 + (13/2)^2}$

$\Delta APC = \frac{\sqrt{3795}}{4}$

②

$\Delta PCQ = \frac{1}{2}|\overrightarrow{QP} \times \overrightarrow{QC}|$

$\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP}$

$\overrightarrow{QP} = \begin{pmatrix} -7 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 19/4 \\ 11/4 \\ 5 \end{pmatrix} = \begin{pmatrix} -9/4 \\ -1/4 \\ 7 \end{pmatrix}$

$\overrightarrow{QC} = \overrightarrow{QO} + \overrightarrow{OC}$

$\overrightarrow{QC} = \begin{pmatrix} -7 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \\ -9 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ -7 \end{pmatrix}$

$\overrightarrow{QP} \times \overrightarrow{QC} = \begin{vmatrix} i & j & k \\ -9/4 & -1/4 & 7 \\ -3 & 4 & -7 \end{vmatrix} = i\begin{vmatrix} -1/4 & 7 \\ 4 & -7 \end{vmatrix} - j\begin{vmatrix} -9/4 & 7 \\ -3 & -7 \end{vmatrix} + k\begin{vmatrix} -9/4 & -1/4 \\ -3 & 4 \end{vmatrix}$

Cross product given in F.B.

$= i\left[(-1/4)(-7) - (4)(7)\right] - j\left[(-9/4)(-7) - (-3)(7)\right] + k\left[(-9/4)(4) - (-3)(-1/4)\right]$

$= -\frac{105}{4}i - \frac{147}{4}j + \frac{39}{4}k$

$\Delta QPC = \frac{1}{2}\left|\begin{pmatrix} -105/4 \\ -147/4 \\ 39/4 \end{pmatrix}\right|$

$\Delta QPC = \frac{1}{2}\sqrt{(-\frac{105}{4})^2 + (-147/4)^2 + (39/4)^2}$

$\Delta QPC = \frac{\sqrt{34155}}{8}$

**(Total for Question 7 is 10 marks)**

$\Delta APC + \Delta QPC = \square APQC$

$= \frac{\sqrt{3795}}{4} + \frac{\sqrt{34155}}{8}$

$= 38.50223208$

$\approx 38.5 \text{ units}^2$ (shown)

b. vol. ABCD $= |\vec{AD} \cdot (\vec{AB} \times \vec{AC})| = 226$

$\vec{AB} = \vec{AO} + \vec{OB}$

$\vec{AB} = \begin{pmatrix}-3\\-4\\-5\end{pmatrix} + \begin{pmatrix}10\\-1\\5\end{pmatrix} = \begin{pmatrix}7\\-5\\0\end{pmatrix}$

$\vec{AC} = \vec{AO} + \vec{OC}$

$\vec{AC} = \begin{pmatrix}-3\\-4\\-5\end{pmatrix} + \begin{pmatrix}4\\7\\-9\end{pmatrix} = \begin{pmatrix}1\\3\\-14\end{pmatrix}$

$\vec{AB} \times \vec{AC} = \begin{pmatrix}7\\-5\\0\end{pmatrix} \times \begin{pmatrix}1\\3\\-14\end{pmatrix} = \begin{array}{ccc} i & j & k \\ \hline 7 & -5 & 0 \\ 1 & 3 & -14 \end{array}$

$= i\begin{vmatrix}-5 & 0\\3 & -14\end{vmatrix} - j\begin{vmatrix}7 & 0\\1 & -14\end{vmatrix} + k\begin{vmatrix}7 & -5\\1 & 3\end{vmatrix}$

$= i[(-5)(-14)-(3)(0)] - j[(7)(-14)-(0)(1)] + k[(7)(3)-(-5)(1)]$

$= 70i + 98j + 26k$

$\vec{AD} = \vec{AO} + \vec{OD}$

$\vec{AD} = \begin{pmatrix}-3\\-4\\-5\end{pmatrix} + \begin{pmatrix}k\\4\\-1\end{pmatrix} = \begin{pmatrix}k-3\\0\\-6\end{pmatrix}$

vol. ABCD: $\left|\begin{pmatrix} k-3 \\ 0 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 70 \\ 98 \\ 26 \end{pmatrix}\right| = 226$

dot product $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = (a)(d) + (b)(e) + (c)(f)$

$\left| 70(k-3) + (0)(98) + (-6)(26) \right| = 226$

$\left| 70k - 210 - 156 \right| = 226$

$\left| 70k - 366 \right| = 226$

(negation)

$70k - 366 = 226$

$70k = 592$

$k = \frac{592}{70} = \frac{296}{35}$

$-(70k - 366) = 226$

$-70k + 366 = 226$

$70k = 140$

$k = 2$

$k = 2$ or $k = \frac{296}{35}$ //

**8.** The hyperbola $H$ has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

The line $l_1$ is the tangent to $H$ at the point $P(4\cosh\theta, 3\sinh\theta)$.

The line $l_1$ meets the $x$-axis at the point $A$.

The line $l_2$ is the tangent to $H$ at the point $(4, 0)$.

The lines $l_1$ and $l_2$ meet at the point $B$ and the midpoint of $AB$ is the point $M$.

(a) Show that, as $\theta$ varies, a Cartesian equation for the locus of $M$ is

$$y^2 = \frac{9(4-x)}{4x} \qquad p < x < q$$

where $p$ and $q$ are values to be determined.

**(11)**

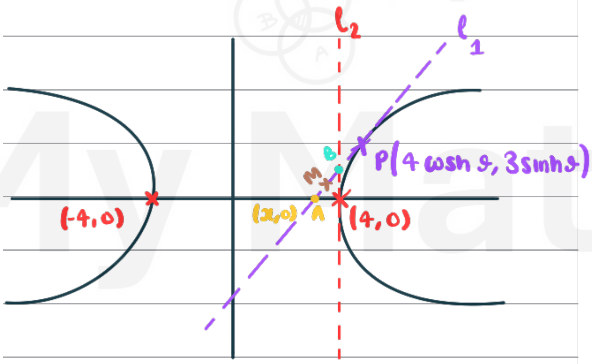
Let $S$ be the focus of $H$ that lies on the positive $x$-axis.

(b) Show that the distance from $M$ to $S$ is greater than 1

**(3)**

can be rewritten as : $\frac{x^2}{(4)^2} - \frac{y^2}{(3)^2} = 1$

$a = 4, \quad b = 3$




**Conics**

| | Ellipse | Parabola | Hyperbola | Rectangular Hyperbola |
|---|---|---|---|---|
| Standard Form | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ | $y^2 = 4ax$ | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $xy = c^2$ |
| Parametric Form | $(a\cos\theta, b\sin\theta)$ | $(at^2, 2at)$ | $(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$ | $\left(ct, \frac{c}{t}\right)$ |
| Eccentricity | $e < 1$ $b^2 = a^2(1 - e^2)$ | $e = 1$ | $e > 1$ $b^2 = a^2(e^2 - 1)$ | $e = \sqrt{2}$ |
| Foci | $(\pm ae, 0)$ | $(a, 0)$ | $(\pm ae, 0)$ | $(\pm\sqrt{2}c, \pm\sqrt{2}c)$ |
| Directrices | $x = \pm\frac{a}{e}$ | $x = -a$ | $x = \pm\frac{a}{e}$ | $x + y = \pm\sqrt{2}c$ |
| Asymptotes | none | none | $\frac{x}{a} = \pm\frac{y}{b}$ | $x = 0, y = 0$ |

Work out $l_1$ first :

$\frac{1}{16}x^2 - \frac{1}{9}y^2 = 1$

$\frac{x}{8} - \frac{2}{9}y\left(\frac{dy}{dx}\right) = 0$ differentiate both sides w.r.t. $x$.

$-\frac{2}{9}y\left(\frac{dy}{dx}\right) = -\frac{x}{8}$

$\frac{dy}{dx} = -\frac{x}{8} \div -\frac{2}{9}y = \frac{x}{8} \times \frac{9}{2y} = \frac{9x}{16y}$

$\frac{dy}{dx} = \frac{9x}{16y}$

**Question 8 continued**

$$\left.\frac{dy}{dx}\right|_{(4\cosh\theta,\,3\sinh\theta)} = \frac{9(4\cosh\theta)}{16(3\sinh\theta)} = \frac{36\cosh\theta}{48\sinh\theta} = \frac{3\cosh\theta}{4\sinh\theta}$$

(sub in P coord to find gradient @ that point)

$$M_{tangent} = \frac{3\cosh\theta}{4\sinh\theta}$$

$$y - 3\sinh\theta = \frac{3\cosh\theta}{4\sinh\theta}(x - 4\cosh\theta)$$

$$4y\sinh\theta - 12\sinh^2\theta = 3\cosh\theta(x - 4\cosh\theta)$$

$$4y\sinh\theta - 12\sinh^2\theta = 3x\cosh\theta - 12\cosh^2\theta$$

$$4y\sinh\theta = 3x\cosh\theta - 12\cosh^2\theta + 12\sinh^2\theta$$

$$4y\sinh\theta = 3x\cosh\theta - 12(\cosh^2\theta - \sinh^2\theta)$$

$\cosh^2\theta - \sinh^2\theta = 1$

$$4y\sinh\theta = 3x\cosh\theta - 12 \quad (l_1)$$

line eq$^n$ $l_2$: $x = 4$

equate $l_1$ and $l_2$ and solve simultaneously.

sub $x = 4$ into $l_1$.

$$4y\sinh\theta = 3(4)\cosh\theta - 12$$

$$4y\sinh\theta = 12\cosh\theta - 12$$

$$y = \frac{12\cosh\theta - 12}{4\sinh\theta} = \frac{12\cosh\theta}{4\sinh\theta} - \frac{12}{4\sinh\theta}$$

$$y = \frac{3\cosh\theta}{\sinh\theta} - \frac{3}{\sinh\theta}$$

Intersection point B: $\left(4, \frac{3\cosh\theta}{\sinh\theta} - \frac{3}{\sinh\theta}\right)$

find coord. A by solving line eq$^n$ and set @ $y = 0$.

$$4y\sinh\theta = 3x\cosh\theta - 12$$

$$\cancel{4(0)\sinh\theta} = 3x\cosh\theta - 12$$

$$3x\cosh\theta = 12$$

$$x = \frac{\cancel{12}\,4}{\cancel{3}_1\cosh\theta} = \frac{4}{\cosh\theta}$$

$$A: \left(\frac{4}{\cosh\theta}, 0\right)$$

Midpoint formulae: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ $(x_1, y_1)$ and $(x_2, y_2)$

$A: \left(\frac{4}{\cosh\theta}, 0\right)$

$B: \left(4, \frac{3\cosh\theta}{\sinh\theta} - \frac{3}{\sinh\theta}\right)$

Midpoint $M: \left(\dfrac{\frac{4}{\cosh\theta} + 4}{2}, \dfrac{0 + \frac{3\cosh\theta}{\sinh\theta} - 3}{2}\right)$

$M: \left(2 + \frac{2}{\cosh\theta}, \frac{3\cosh\theta}{2\sinh\theta} - \frac{3}{2\sinh\theta}\right)$

WORKING OUT P & Q

Let $X = 2 + \frac{2}{\cosh\theta}$

$X - 2 = \frac{2}{\cosh\theta}$

$\cosh\theta = \frac{2}{x-2}$

largest value $\cosh\theta$ can be is $\infty$.

$\therefore x = 2 + \frac{2}{\cosh\theta}$

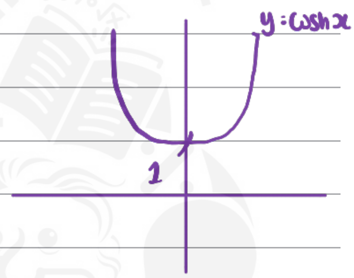
where $\cosh\theta \to \infty$, $2 + 0 \approx 2$.

$x > 2$

Smallest value $\cosh\theta$ can be is $1$.

$\therefore x = 2 + \frac{2}{1} = 4$

$\therefore x < 4$




Let $y = \frac{3\cosh\theta}{2\sinh\theta} - \frac{3}{2\sinh\theta}$

$y^2 = \left(\frac{3\cosh\theta}{2\sinh\theta} - \frac{3}{2\sinh\theta}\right)^2 = \frac{(3\cosh\theta - 3)^2}{(2\sinh\theta)^2} = \frac{(3\cosh\theta - 3)^2}{4\sinh^2\theta}$

Change $\sinh^2\theta$ to $\cosh^2\theta$ @ denominator

$y^2 = \frac{(3\cosh\theta - 3)^2}{4(\cosh^2\theta - 1)}$

Said earlier $\cosh\theta = \frac{2}{x-2}$

$y^2 = \frac{\left(3\left(\frac{2}{x-2}\right) - 3\right)^2}{4\left(\left(\frac{2}{x-2}\right)^2 - 1\right)} = \frac{\left(\frac{6}{x-2} - 3\right)^2}{4\left(\frac{4}{(x-2)^2} - 1\right)}$



P 6 1 1 7 9 A 0 3 0 3 2

Question 8 continued

$$y^2 = \frac{\left(\frac{6}{x-2} - 3\right)^2}{4\left(\frac{4}{(x-2)^2} - 1\right)}$$

$$y^2 = \frac{\left[3\left(\frac{2}{x-2} - 1\right)\right]^2}{4\left(\left(\frac{2}{x-2}\right)^2 - 1\right)}$$

can be split into difference of 2 squares

$$y^2 = \frac{9\left(\frac{2}{x-2} - 1\right)^{\cancel{2}}}{4\cancel{\left(\frac{2}{x-2} - 1\right)}\left(\frac{2}{x-2} + 1\right)}$$

$$y^2 = \frac{9\left(\frac{2}{x-2} - 1\right)}{4\left(\frac{2}{x-2} + 1\right)}$$

multiply numerator and denominator by $(x-2)$ to remove awkward $(x-2)$ denominator

$$y^2 = \frac{9(x-2)\left(\frac{2}{x-2} - 1\right)}{4(x-2)\left(\frac{2}{x-2} + 1\right)}$$

$$y^2 = \frac{9(2 - (x-2))}{4(2 + (x-2))}$$

$$y^2 = \frac{9(2 - x + 2)}{4(2 + x - 2)}$$

$$y^2 = \frac{9(4-x)}{4x} \quad //$$ (shown)

$2 < x < 4$ $\quad p = 2 \quad q = 4$

**Question 8 continued**

b. must first work out focus of H:

$(\pm ae, 0)$

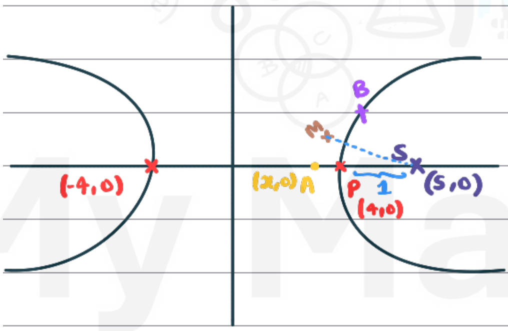
Must work out $e$.

$b^2 = a^2(e^2-1)$
$(3)^2 = (4)^2(e^2-1)$
$9 = 16(e^2-1)$
$\frac{9}{16} = e^2 - 1$
$e^2 = \frac{25}{16}$
$e = \pm\sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

$e > 1$
$\therefore e = +\frac{5}{4}$

$(\pm(4)(\frac{5}{4}), 0)$

$S: (\pm 5, 0)$ foci of H.




distance $|PS| = 5 - 4 = 1$
M is further away from P than 1.
$\therefore |MS| > 1$ //

(Total for Question 8 is 14 marks)

**TOTAL FOR PAPER IS 75 MARKS**